Circle of Confusion and the Thin Lens Equation

Introduction

The question of what is in focus is subjective. When we look at an image, parts of it will appear in focus and parts will not. The parts that appear acceptably sharp are covered by a distance called the *depth of field*, even though there is only one plane in front of the camera that is technically in focus. The farther away an object is from that plane, the more out of focus it will appear. A point on the plane of focus will appear as point on the sensor or film. However, if that point is moved forwards or backwards from the plane of focus it will appear as a disk, or a blur spot. For small distances from the plane of focus, a typical human observer will not notice a difference between the point and the disk. The largest possible disk that appears still in focus is called the *circle of confusion* c.

Circle of Confusion

The circle of confusion is the largest blur spot that will be perceived by the human eye as a point; it depends on how much the image out of camera will be magnified. If the film or sensor image is magnified by a lot, for objects to appear sharp *c* will have to be small, and vice versa.

In order to compute *c* some *subjective* assumptions must be made. Typically, the following assumptions are made:

1. The final image is of size 8in by 10in will be viewed from distance of 25cm; a person with good vision will be able to resolve 5 lines per mm at that distance
2. We took the photograph on an 8x10 camera and there is no magnification when the final image is viewed

Therefore, we have *c* diameter of .2 mm for the 8x10 camera. If we took the same image on a 35mm film (24mm x 36mm), we require a magnification of about eight times to print the photograph. Thus, for the 35mm camera, *c* would be .2/8mm or about .025mm; points focused to a disk that is smaller than .025mm on the film will appear to be in focus.

It is convenient to relate *c* to the diagonal of the sensor. For a 35mm sensor with a diagonal of about 43.25mm, and *c* of .025mm, in order to get .025mm we must divide 43.25 by 1730. We thus arrive at the formula

$$c = \frac{diag}{1730}$$

where *diag* is the diagonal of film or sensor. Clearly, the said assumptions can be changed based on expected magnification of the final image and the viewing distance. For example, if we are viewing a digital image on a computer, and we magnify the image to 100% magnification so that we can see individual pixels, the circle of confusion should perhaps be defined to four pixels across. Suppose that in this example the cameras resolution is 8256x5504, for a total of about 45M pixels; the diameter of c would be two pixels or .017mm. In this case we would use

$$c = \frac{diag}{2544}$$

In our analysis below we will 2544 as the denominator. The argument in support of this is that most photographers will check whether a photograph is in focus by zooming all the way in. At that point the circle of confusion would be no more than four pixels diagonally.

As the circle of confusion is entirely subjective, one can argue that the higher the resolution of a sensor the more magnification an image will allow; and thus the depth of field will decrease with sensor resolution.

Thin Lens Equation

Focal length is a measure of how strongly the lens converges or diverges. When the lens itself is approximated by a thin lens, and when the object is at infinity, the lens will create an image of the object at the focal length behind the lens. In order to focus a camera where the film is at a fixed position in the camera, the lens (thin lens) must move. The distance to the object is related to the distance and the focal length as follows:

$$1/_o + 1/_i = 1/_f$$

Where *o* is the object distance, *i* is the image distance, and *f* is the focal length.

We will use this equation to find equivalent focusing distance for lenses of different focal lengths such that the size of image on the sensor is the same height. For example, we may want to find the equivalent object distance on a 50mm lens given that we have focused an object from two meters on a 105mm lens. We want to solve for *o* given height of image on film h_i, real life height of object h_o, and *f*.

Since

$$\frac{h_o}{o} = \frac{h_i}{i}$$

we have two equations in two unknowns. After a few lines of arithmetic

$$o = f + \frac{f}{h_i/_{h_o}}$$

Since *f* is usually much smaller than *o,* we can safely use the following approximation:

$$\frac{o}{f} = \frac{h_o}{h_i}$$

Therefore, if we are going to maintain the same composition, when we increase the focal length, we must increase the distance to object by the same factor to keep the image in focus. Suppose we have an object that is 1000mm in height and it generates an image of 30mm, it is in focus at 2m away from the camera, and the focal length is 50mm. If we want to change lenses to 100mm and keep the image at 30mm in size, we need to increase distance to camera to 4m.

Depth of Field

It is intuitive to think of depth of field in terms of the circle of confusion as depth of field is determined by it. Most importantly, depth of field is proportional to diagonal of the sensor. All else being equal, a smaller sensor will have a smaller depth of field. However, in practice this rarely the case given that other factors are involved in computing the depth of field.

Aperture and Depth of Field

Consider a point in front of the camera that is not on the plane of perfect focus. As before, this point will generate a blur spot on the film. The size of this blur spot is dependent on the aperture. The smaller the aperture the smaller that blur will be and will approximate a point more closely. Therefore, stopping down, or increasing aperture number, causes more points to fall within c and thus increases the depth of field.

Distance to subject and Depth of Field

Greater distance of subject to lens affords less variability in points from the subject that appear on the sensor. Therefore, when the subject is far away and in focus, the subject can move back and forth and remain mostly in focus. When the subject is close, even a small movement can make it appear out of focus. In other words, the greater the distance the less magnified the subject is. As an approximation depth of field increases with the square of distance to the plane in focus.

Focal length and Depth of Field

For a single lens, focal length is defined as distance between film and the center of the lens. Increasing the focal length causes more of the subject to fill the frame. As the subject is magnified smaller portions of the subject can fall in the circle of confusion decreasing the depth of field. As an approximation, depth of field is inversely proportional to the square of focal length.

Depth of field DOF Formula

The formula for the depth of field is a direct consequence of the thin lens equation. Depth of field is not the same in front of the plane of focus as it is in the back. Depth of field is therefore a sum of the front and back distances:

$$d = d_{front} + d_{back}$$

Starting with the thin lens equation it is possible to derive the following two equations:

$$d_{front} = \frac{nco^2}{f^2 + nco}$$

$$d_{back} = \frac{nco^2}{f^2 - nco}$$

Where n is the f-stop number set on the lens, c is the circle of confusion, o is the distance to object, and f is the focal length.

It is sometimes convenient to express the total depth of field using an approximation where the nco term is assumed to be small:

$$d_{total} = \frac{2nco^2}{f^2}$$

Note that depth of field behind the plane of focus is not the same as it is in front. In *Table 1*, the object distance *o* is 5m for the 50mm lens, and f-stop number is 1.4. Similarly, for the 200mm lens, the object distance *o* would be 20m. In this way, depth of field is computed for equal image size on the sensor for each focal length. In other words, we change the lens and recompose to the same original composition.

f	d_front	d_back	d_total
35mm	223	255	478
50mm	227	250	477
85mm	232	245	477
105mm	233	244	476
135mm	234	242	476
200mm	235	241	476

Table 1: Focal length versus depth of field (in mm) for equal image size at f-stop=1.4 and distance to object 5m for the 50mm lens.

Table 1 shows that for wide angle lenses, the back depth of field tends to be larger than the front. For longer lenses the situation is reversed. For a given f-stop and object distance, a short focal length lens will have a larger back depth of field than a long lens, even though the total depth of field will be nearly the same. In other words, long lens will defocus far away objects faster than a wide-angle lens.

As show in *Table 1*, if we keep the image size on the sensor constant, total depth of field will not be affected by focal length. Thus, when comparing lenses of different focal lengths, the most important consideration is the maximum f-stop of the lens and not its focal length. However, when comparing the quality of the defocused area, or what is known as bokeh, the focal length will play a significant role due to image compression.

Depth of field and Bokeh

If we compare two lenses, with the same maximum aperture, they will both have the same depth of field when wide open. However, the longer lens will exhibit a smoother bokeh. There are two reasons:

- The long lens will have a narrower field of view and will capture a smaller portion of the background
- Objects behind the plane of focus are defocused faster using a long lens than a short lens

A longer lens, used at a longer distance, and shot at a larger f-stop number, may therefore appear as if it has a shallower depth of field than shorter lens at a lower f-stop.

Perspective distortion and focal length

As most photographers have noticed, wide angle lenses have the property where objects that are close to the camera appear enlarged. Yet again, perspective distortion is a direct consequence of the thin lens equation. Recall that

$$\frac{o}{f} = \frac{h_o}{h_i}$$

Therefore

$$h_i = \frac{h_o}{o} f$$

and differentiating with respect to *o* we obtain

$$\frac{\partial h_i}{\partial o} = -\frac{h_o}{o^2} f$$

The formula explains the distortion we see when taking a picture at small object distances. When o is small, $\frac{\partial h_i}{\partial o}$ is larger in magnitude, this means that small distances are exaggerated. Thus, even though the tip of the nose in a portrait is only a few centimeters closer to the camera, its size will be enlarged by $1/_{o^2}$ which will be a large number when *o* is small. This effect is commonly referred to as distortion. Now when *o* is large, $\frac{\partial h_i}{\partial o}$ will approach zero. Therefore, objects far away do not change size with distance. If two objects are far away, it will be impossible to tell which one is closer are closer to the camera if their distance from camera is sufficiently large. This effect is referred to as compression.

Since wide angle lenses tend to be used at small distances for portrait photography, we tend to think of wide-angle lenses as having distortion. As matter of fact, as $\frac{\partial h_i}{\partial o}$ is proportional to focal length, it is long focal length lenses that have more distortion.

Long focal length lenses are thought of as exhibiting compression as they tend to be used at longer object distances for portrait photography.

Creating equivalent image for different format sizes

Equivalent angle of view and focal length

A reasonable assumption to computing the angle of view at infinity, is

$$\emptyset = \frac{diag}{f}\frac{180}{\pi}$$

Where *diag* is the diagonal and *f* and is the focal length.

Therefore, as diagonal of the sensor increases, to maintain the same angle of view, focal length must increase proportionately.

To find the equivalent focal length for sensors with different diagonals we use

$$\frac{diag_1}{f_1} = \frac{diag_2}{f_2}$$

and multiply the focal length by the ratio of diagonals.

Equivalent f-stop number

In order to get equivalent depth of field from the two lenses we must have

$$d_1 = d_2$$

expanding

$$\frac{n_1}{n_2} = \frac{f_1^2}{f_2^2}\frac{o_2^2}{o_1^2}\frac{diag_2}{diag_1}$$

and therefore

$$\frac{n_1}{n_2}\frac{diag_1}{diag_2} = \left(\frac{f_1 o_2}{f_2 o_1}\right)^2 = \left(\frac{diag_1}{diag_2}\right)^2$$

Thus, we have

$$\frac{n_1}{n_2} = \frac{diag_1}{diag_2}$$

Thus, to achieve equivalent depth of field from both focal lengths we simply set their f-stop numbers to the ratio of diagonals.

Equivalent aperture

The lens aperture ω, as specified by the f-number, is the ratio of focal length to the diameter of opening created by the lens diaphragm. As the focal length is closely related to the distance light must travel through the lens to reach the sensor, aperture gives us a measure of amount of light hitting the sensor. Intensity of light arriving at the sensor is inversely proportional to the that distance. Thus, given the radius *r* of diaphragm, aperture is given by

$$\omega = \frac{f}{2r}$$

Amount of light hitting the sensor, or intensity *I*, is proportional to

$$I \propto \frac{2\pi r^2}{f^2} = \frac{\pi}{2\omega^2}$$

Assume that we are using a lens whose maximum aperture is ${}^{f}/_{r} = 1$. If we want to stop down the lens and decrease intensity by a factor of 2. To do this we need to multiply ω by $\sqrt{2}$. Therefore, successive f-stops numbers starting with 1 correspond to 1, $\sqrt{2}$,2,2$\sqrt{2}$,4,4$\sqrt{2}$ and so on. In decimal this translates to the familiar 1,1.4,2,2.8,4,5.6 etc.

Thus, when working to achieve an equivalent look from different formats we will need to adjust the f-stop number to maintain equivalence in depth of field. This will change the light intensity at the film or senor. We can simply adjust for this effect by multiplying exposure or sensor sensitivity by the diagonal ratio.

Replicating the Film Medium Format Look

In order to discuss a subjective topic such as the so called "Medium Format Look" we must first define it. As matter of opinion, the three most iconic cameras of the era are the Hasselblad 501, Pentax 6x7, and Mamiya RZ67. As far as I am aware the Pentax Takumar 105mm f2.4 is the lens with the smallest maximum aperture. It is therefore the hardest to replicate. I think it is therefore reasonable to define the medium format look in terms of the look of this lens.

Table 2 below shows the equivalent focal length and maximum aperture required to match the angle of view as well as the depth of field of the Takumar 105mm f2.4 in other formats. In the full frame format, we would need a 49mm lens with a f1.1 maximum aperture. There will be no difference in composition or perspective.

Format	Diag ratio	f	f-stop	dof front	dof back
6x7	1	105mm	f2.4	31mm	32mm
6x6	0.92	97mm	f2.2	31mm	32mm
6HD 100	0.72	76mm	f1.7	31mm	32mm
Crop MF	0.59	62mm	f1.4	31mm	32mm
FF	0.47	49mm	f1.1	31mm	32mm
APS-C	0.31	32mm	f0.7	31mm	32mm
m4/3	0.23	25mm	f0.6	31mm	32mm

Table 2: Equivalent focal length and maximum aperture as compared to Pentax 6x7 Takumar 105 f2.4. Depth of field front and back calculated at distance of 2m.

The main issues with replicating the medium format look is availably of lenses with low enough f-stop number. For example, in the micro four thirds and the APS-C formats it is not possible to find such lenses. In the full frame format the situation is a bit better because we could there a few f1.2 lenses available, there are also sub f1.0 lenses available from Nikon, Leica, and Mitakon. In the cropped digital medium format, a well as the full frame medium format, there really are few if any viable options in terms of matching the depth of field of the Takumar.